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Some shape optimization problems for eigenvalues

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Abstract
In this work we consider some inverse problems with respect to domain for the Laplace operator. The considered problems are reduced to the variational formulation. The equivalency of these problems is obtained under some conditions. The formula is obtained for the eigenvalue in the optimal domain.

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1. Introduction

The study of shape optimization problems is a very wide field of optimization theory. It begins with classical problems, as the isoperimetric and the Newton problem of the best aerodynamical shape show. Recent results have also been obtained in the last three decades.

This kind of problem for the eigenvalues of an elliptic operator is an intensively studied field that has strong relations with several applications such as, for instance, the stability of vibrating bodies, the propagation of waves in composite media, and the thermic insulation of conductors. Some characteristics of these systems are described by the eigenvalues of the corresponding operators. For instance, the eigenvalues of the Schrödinger operator $Lu(x) = -\Delta u(x) + q(x)u(x)$ are energy levels of the quantum particle in the external force field [1], of the operators $Lu(x) = -\Delta u(x)$ and $Lu(x) = \Delta^2 u(x)$—eigenfrequency of the vibrating membrane and plate, correspondingly [2]. Investigation of such problems is also important in studying qualitative properties of the eigenvalues.

The fascinating feature is that the objects under investigation are shapes, i.e. domains of R^m, instead of functions, as usually occurs in the problems of variational calculus. This constraint often produces additional difficulties that lead to a lack of existence of a solution and to the introduction of suitable relaxed formulation of the problem. However, in some cases an optimal solution exists, due to the special form of the cost functional and geometrical restrictions on the class of admissible domains [3]. Another difficulty is related mainly to the mathematical definition of the variation of the domain characterized by the variation of its boundary. Introduced by J Cea and developed by J Sokolowski, J-P Zolesio and other researchers, the methods of variation of the domain using vector fields allowed one to solve some of these problems [4]. But these techniques encounter some difficulties from theoretical and numerical points of view [5]. They often require an initialization close to the optimum and in that can prove to be inapt to identify a non-intuitive optimal solution; the calculation of the gradient is always delicate, seldom exact and generally expensive [6]. Moreover, the following problems arise.

To connect the set of the admissible domains to the vector fields; to put a high regularity of the initial data; to solve the conditional optimization problem by this method, it is usually necessary, to reduce it to a non-conditional problem (for example, by the Lagrange multipliers technique).

The new approach introduced in [7, 8] tends to avoid these difficulties. It consists of representing a convex domain (or a pair of domains) by its support function . It is shown that the set of such domains forms a structure of linear space and one can even define a scalar product in it.

The variation of the domain then is naturally replaced by the variation of the corresponding support function. For any convex-bounded domain its support function is continuous convex and positive homogeneous. Also it is known that for each continuous convex positive-homogeneous function there exists a convex bounded set, such that this function is a support function for this set. The set coincides with the sub-differential of this function at the origin [9]. This single-valued correspondence between domains and convex and positive homogeneous functions allows us to express the variation of the domain by the variation of the corresponding support function.

In the process of numerical simulation after each iteration we not only get a set of boundary points, but also a support function. The domain is reconstructed as a sub-differential of its support function in the point 0. It allows one to avoid the necessity to control: Does the set of boundary points form a shape of domain or not?

Based on this technique we proved the differentiability of the eigenvalue of the elliptic operator with respect to domain and got a formula for its first variation [10]. The results obtained in this work are mainly based on these formulae.

In this work we consider some inverse problems relatively domain. These problems are reduced to the shape optimization problems for the functionals related to the eigenvalues of the elliptic operators. Here we show that under some conditions these problems are equivalent and get a formula for the eigenvalue in the optimal domain. Note that the obtained formulae do not include an eigenfunction. This fact makes them interesting both from practical and theoretical points of view.

For the sake of simplicity we consider only a Laplace operator, but the results may be extended for other elliptic operators.

2. Main results

Consider the problem

$$-\Delta u = \lambda u, \qquad x \in D, \tag{1}$$

$$u(x) = 0, \qquad x \in S_D, \tag{2}$$

where Δ is the Laplace operator, $D \subset R^m$—a convex bounded domain, S_D—its boundary and $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$.

It is known [1] that in the considered case the eigenfunctions (eigenvibrations of the membrane) u_j, $j = 1, 2, \ldots$ of the problem (1), (2) belong to the class $C^2(D) \cap C^1(\overline{D})$, eigenvalues (eigenfrequencies) are positive and may be numbered as $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$, where each λ_k is counted with its multiplicity.

Let $D \subset R^m$ be a bounded convex domain with a smooth boundary S_D. We denote the set of all such domains by K.

For the sake of simplicity we denote by $u(x)$ the first normalized eigenfunction of the problem (1), (2) corresponding to the first eigenvalue λ_1.

First we consider the problem: given $\varphi(x) \in C^1(R^m)$, define $D \in K$ such that the relation

$$\frac{|\nabla u(x)|^2}{\lambda_1} = \varphi(x), \qquad x \in S_D \tag{3}$$

is valid for the first eigenvalue and the corresponding eigenfunction of the problem (1), (2).

We call this $\mathcal{I}$ problem.

Parallely let us consider the following variational problem:

$$-\Delta u = \lambda u, \qquad x \in D, \tag{4}$$

$$u(x) = 0, \qquad x \in S_D, \tag{5}$$

$$\lambda_1(D) \to \min, \tag{6}$$

under the condition

$$\int_D f(x)\,\mathrm{d}x = 1, \tag{7}$$

where $f(x)$ is given in the R^m function.

This problem we call $\mathcal{V}$ problem.

Here we give a theorem that plays an important role in the investigation of these problems.

Theorem 1. *For the eigenvalue of the problem (1), (2) in the domain D the following formula is valid:*

$$\lambda_1 = \frac{1}{2}\int_{S_D} |\nabla u(x)|^2 P_D(n(x))\,\mathrm{d}s, \tag{8}$$

where $P_D(x) = \max_{l \in D}(l, x)$, $x \in R^m$ *is a support function of the domain* D, $n(x)$ *is an outward normal to* S_D *in the point* x.

It needs to be noted that the formula (8) is true for all eigenvalues of the Schrödinger operator when the potential $q(x)$ *is* -2*-order homogeneous, i.e.*

$$q(tx) = \frac{1}{t^2}q(x).$$

This theorem shows that boundary values of the function $|\nabla u_j(x)|$ *uniquely define eigenfunction* λ_j. *Formula (8) and this fact may be attractive in the spectral theory of the operators.*

As noted above, solution of the inverse problem relatively domain meets some serious difficulties. One of the methods to their solution is reducing of such problems to the variational ones (shape optimization problems). The theorem below puts a relation between the solutions of the $\mathcal{I}$ *and* $\mathcal{V}$ *problems.*

Theorem 2. *Let* $\varphi(x)$ *be an* α*-order homogeneous function, i.e.*

$$\varphi(tx) = t^{\alpha}\varphi(x), \tag{9}$$

$f(x) = C\varphi(x)$, where $C = \frac{\alpha+m}{2}$. Then the solution of the $\mathcal{V}$ problem will be the solution of the $\mathcal{I}$ problem at the same time.

As an example to the functions satisfying the condition of the theorem take

$$\varphi(x) = x_1^\alpha \Phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_m}{x_1}\right), \tag{10}$$

where Φ is any differentiable function.

Then the function

$$f(x) = \frac{m+\alpha}{2} \cdot \varphi(x)$$

satisfies the condition of the theorem.

Now suppose that $\varphi(x) \equiv 1$. In this case $\alpha = 0$ and

$$f(x) = \frac{m}{2}.$$

For this case condition (7) takes the form

$$\text{mes}\, D = \frac{2}{m}.$$

Note that this condition occurs during the formulation of some practical problems.

To solve the $\mathcal{V}$ problem one can introduce a Lagrange multiplier and reduce the problem to the non-conditional case. In this case the solution of the considered problem relates the following shape optimization problem:

Given $f(x) \in C^1(R^m)$, define $D \in K$ such that

$$J(D) = \lambda_1(D) + \int_D f(x)\, \mathrm{d}x \to \min. \tag{11}$$

An important problem is the estimation of the mechanical characteristics of the system in the optimal domain. It may have significant applications to the solution of some practical problems. The next theorem allows one to calculate the eigenfrequency of the membrane during across vibrations. The attraction of the obtained formula is that it does not contain an eigenfunction.

Theorem 3. *If the domain $D \in K$ is a solution of the problem (1), (2), (11) then*

$$\lambda_1(D) = \frac{1}{2}\int_{S_D} f(x) P_D(n(x))\, \mathrm{d}s. \tag{12}$$

Let us consider some particular cases. Suppose that $f(x) = 1, x \in R^m$. In this case problem (11) takes the form

$$J(D) = \lambda_1(D) + \text{mes}\, D \to \min. \tag{13}$$

From (12) we obtain

$$\lambda_1(D) = \frac{1}{2}\int_{S_D} P_D(n(x))\, \mathrm{d}s.$$

As is known ([7])in the two-dimensional case,

$$\frac{1}{2}\int_{S_D} P_D(n(x))\, \mathrm{d}s = \text{mes}\, D.$$

Thus

$$\lambda_1(D) = \text{mes}\, D.$$

This formula shows that in the two-dimensional case the eigenvalue (eigenfrequency) corresponding to the domain, that gives a minimum to the functional (11) is equal to the area of this domain (domain of the membrane).

In the one-dimensional case problem (1), (2), (11) turns to

$$u'' = \lambda u, \qquad x \in (a, b),$$
$$u(a) = u(b) = 0,$$
$$\lambda_1(a, b) + \int_a^b f(x)\,\mathrm{d}x \to \min.$$

For this problem as one may obtain from (12)

$$\lambda_1(a, b) = \tfrac{1}{2}[f(b)b - f(a)a].$$

Other cases may also be considered when the inverse problem (1)–(3) is reduced to the variational formulation (4)–(7) with various functionals. To investigate the problem (4)–(7) one may use the apparatus offered in [1].

3. Proof of theorem 1

It is known that [11] for a fixed domain D the first eigenvalue of the problem (1), (2) is calculated by the formula

$$\lambda_1(D) = \inf_u I(u, D),$$

where

$$I(u, D) = \frac{\int_D |\nabla u(x)|^2\,\mathrm{d}x}{\int_D u^2(x)\,\mathrm{d}x}, \qquad |\nabla u(x)|^2 = \sum_{i=1}^m \left(\frac{\partial u}{\partial x_i}\right)^2$$

and inf is taken over all functions $u \in C^2(D) \cap C^1(\bar{D})$, being equal to zero on S_D.

As we see, this formula defines λ_1 as a functional of D.

In [7, 10] it is proved that the first eigenvalue of the problem (1), (2) is differentiable with respect to D on K and for its first variation the formula

$$\delta\lambda_1(D) = -\int_{S_D} |\nabla u|^2 \delta P_D(n(x))\,\mathrm{d}s \tag{14}$$

is valid.

Let us take a positive parameter t and define $D = D(t)$, $\lambda_1(t) = \lambda_1(D(t))$. Then as one may obtain from (14)

$$\lambda_1(t + \Delta t) - \lambda_1(t) = \lambda_1(D(t + \Delta t)) - \lambda_1(D(t))$$
$$= \int_{S(t)} |\nabla u_1(x)|^2 [P_{D(t+\Delta t)}(n(x)) - P_{D(t)}(n(x))]\,\mathrm{d}s + o(\Delta t).$$

From this dividing by Δt we obtain

$$\lambda_1'(t) = -\int_{S_{D(t)}} |\nabla u_1(x)|^2 P_{D(t)}'(n(x))\,\mathrm{d}s, \tag{15}$$

where

$$P_{D(t)}'(x) = \frac{\mathrm{d}}{\mathrm{d}t} P_{D(t)}(x).$$

Now let $D_0 \in K$, $D(t) = tD_0$, $t > 0$. Then

$$-\Delta u(x) = \lambda_1(D_0)u(x), \qquad x \in D_0.$$

One can write this equation in the following equivalent form

$$-\frac{1}{t^2}\Delta_{(\frac{x}{t})}u\left(\frac{x}{t}\right) = \frac{\lambda_1(D_0)u}{t^2}\left(\frac{x}{t}\right), \qquad x \in D(t). \tag{16}$$

From this it is clear that the function $\tilde{u}(x) = u\left(\frac{x}{t}\right)$, $x \in D(t)$ would be an eigenfunction of the problem (1), (2) by $D = D(t)$.

Really, as for the function

$$\tilde{u}(x) = u\left(\frac{x}{t}\right),$$

the relation

$$\Delta\tilde{u}(x) = \frac{1}{t^2}\Delta u\left(\frac{x}{t}\right), \qquad x \in D(t)$$

is true, from (14) we obtain

$$-\Delta\tilde{u}(x) = \frac{\lambda_1(D_0)}{t^2}u(x).$$

It shows that $\tilde{u}(x)$ is an eigenfunction of the problem (1), (2) corresponding to the eigenvalue $\lambda_1(t) = \frac{\lambda_1(D_0)}{t^2}$. Considering this in (15) we have

$$-2\frac{\lambda_1(D_0)}{t^3} = -\frac{1}{t^2}\int_{S_D}\left|\nabla u\left(\frac{x}{t}\right)\right|^2 P_{D_0}(n(x))\,\mathrm{d}s.$$

Taking $t = 1$ we obtain (8).

The theorem is proved.

4. Proof of theorem 2

To prove the theorem let us introduce Lagrange's function

$$L(D, \mu) = \lambda_1(D) + \mu \cdot \int_D f(x)\,\mathrm{d}x. \tag{17}$$

Denote

$$G(D) \equiv \int_D f(x)\,\mathrm{d}x.$$

In [7], it is shown that this functional is differentiable on K and

$$\delta G(D) = \int_{S_D} f(x)\delta P_D(n(x))\,\mathrm{d}s. \tag{18}$$

Considering this and (14) in (17) we obtain

$$\delta L(D) = \int_{S_D}[\mu f(x) - |\nabla u|^2]\delta P_D(n(x))\,\mathrm{d}s. \tag{19}$$

If D is a solution of the problem (4)–(7) then as follows from Lagrange's theory the equality

$$\mu \cdot f(x) - |\nabla u|^2 = 0, \qquad x \in S_D \tag{20}$$

is satisfied.

In spite of this relation being obtained under the convexity condition on D, one may show that it is true for non-convex domains too.

Multiplying (20) by $P_D(x)$ and integrating on S_D we obtain

$$\int_{S_D} [\mu \cdot f(x) - |\nabla u|^2] P_D(n(x))\,\mathrm{d}s = 0.$$

From this it is easy to obtain

$$\frac{1}{2}\mu \cdot \int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s = \frac{1}{2}\int_{S_D} |\nabla u|^2 P_D(n(x))\,\mathrm{d}s.$$

The right-hand side of this equality is equal to the first eigenvalue of the problem (1)–(2) (see theorem 1).

Considering this we obtain

$$\mu \cdot \int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s = 2\lambda_1.$$

Since the first eigenvalue of the problem (1), (2) is not equal to zero,

$$\int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s \neq 0.$$

So we obtain for Lagrange's multiplier,

$$\mu = \frac{2\lambda_1}{\int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s}. \tag{21}$$

This formula allows one to calculate the Lagrange multiplier. Note that here D is an optimal domain and (21) does not include an eigenfunction.

Putting (21) into (20) we obtain

$$\frac{2\lambda_1 \cdot f(x)}{\int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s} = |\nabla u|^2. \tag{22}$$

From this we finally obtain

$$\frac{|\nabla u|^2}{\lambda_1} = \frac{2f(x)}{\int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s}. \tag{23}$$

The proof will be finished if we show the validity of the following equality:

$$\frac{2f(x)}{\int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s} = \varphi(x), \qquad x \in S_D. \tag{24}$$

Denote

$$M = \int_{S_D} f(x) P_D(n(x))\,\mathrm{d}s.$$

As is known [9] for the support function $P_D(x)$,

$$P_D(n(x)) = (n(x), x), \qquad x \in R^m \tag{25}$$

is true.

Using this and the Gauss–Ostrogradskii formula we can write

$$\begin{aligned} M &= \int_{S_D} f(x)(n(x), x)\,\mathrm{d}s = \sum_{i=1}^{m} \int_D \frac{\partial}{\partial x_i}(f(x), x)\,\mathrm{d}x \\ &= \sum_{i=1}^{m} \int_D \left(f(x) + \frac{\partial f(x)}{\partial x_i} x_i \right) \mathrm{d}x \\ &= \int_D [m \cdot f(x) + (\nabla f, x)]\,\mathrm{d}x. \end{aligned}$$

As follows from condition (7),

$$M = m + \int_D (\nabla f, x)\,\mathrm{d}x. \tag{26}$$

Differentiating (9) with respect to t and then taking $t = 1$, it is easy to check that for the functions $\varphi(x)$ is true

$$(\nabla\varphi(x), x) = \alpha\varphi(x).$$

Considering this and condition (7) one may obtain

$$M = m + C\int_D (\nabla\varphi(x), x)\,\mathrm{d}x = m + \alpha.$$

Thus

$$\frac{2f(x)}{M} = \frac{2C\varphi(x)}{m+\alpha}.$$

As $C = \frac{m+\alpha}{2}$, from this we obtain (24).

The theorem is proved.

5. Proof of theorem 3

Considering (14) and (18) we can write

$$\delta J(D) = \delta\lambda_1(D) + \delta\int_D f(x)\,\mathrm{d}x = \int_{S_D} [f(x) - |\nabla u|^2]\delta P_D(n(x))\,\mathrm{d}s. \tag{27}$$

Now, let $D \in K$ be a solution of the problem (1), (2), (11). Then according to the optimality condition,

$$-|\nabla u(x)|^2 + f(x) = 0, \qquad x \in S_D. \tag{28}$$

Multiplying (28) by $P_D(n(x))$ and integrating over S_D we obtain

$$-\frac{1}{2}\int_{S_D} |\nabla u(x)|^2 P_D(n(x))\,\mathrm{d}s + \frac{1}{2}\int_{S_D} f(x)P_D(n(x))\,\mathrm{d}s = 0.$$

Considering here (8) we obtain the statement of the theorem.

The theorem is proved.

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References

[1] Vladimirov V S 1988 *Equations of Mathematical Physics* (Moscow: Nauka)
[2] Gould S H 1996 *Variational Methods for Eigenvalue Problems* (Oxford/Toronto, Canada: Oxford University Press/University of Toronto Press)
[3] Bugur D, Buttazzo G and Figueiredo I 1999 On the attainable eigenvalues of the Laplace operator *SIAM J. Math. Anal.* **30** 527–36
[4] Zolesio J-P 1981 Domain vibration formulas for free boundary problem *Optimization of Distributed Parameter Systems* vol II (Amsterdam: Nijhoff, Nagae) pp 1152–94
[5] Kerimov A K 1982 On Free Boundary Optimization Problems *Dokl. Akad. Nauk SSSR* **266** 545–8
[6] Chakib A and Nachaoui A 2005 Nonlinear programming approach for a transient free boundary flow problem *Appl. Math. Comput.* **160** 317–28

[7] Niftiyev A A and Gasimov Y S 2004 *Control by Boundaries and Eigenvalue Problems with Variable Domains* (Baku: Publishing House of Baku State University)

[8] Gasimov Y S and Niftiyev A A 2001 On a minimization of the eigenvalues of Schrödinger operator over domains *Dokl. RAS* **380** 305–7

[9] Demyanov V F and Rubinov A M 1990 *Basises of Non-Smooth Analyses and Quazidifferential Calculas* (Moscow: Nauka)

[10] Gasimov Y S 2003 On some properties of the eigenvalues when the domain varies *Math. Phys. Anal. Geom.* **10** 249–55

[11] Mikhailov V P 1976 *Partial Differential Equations* (Moscow: Nauka)